Announcements

1) HW 4 up, due next week

2) One last candidate tomorrow
   2-3 CB 2070
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Theorem: If $X = \mathbb{R}$ with the metric $d(x,y) = |x-y|$, then $\mathbb{R}$ is complete.

proof: Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$

We know from last class that all Cauchy sequences are bounded.

By Bolzano-Weierstrass,

the sequence $(x_n)_{n \in \mathbb{N}}$

admits a convergent

subsequence $(x_{n_k})_{k \in \mathbb{N}}$.

Call the limit of the

subsequence L.

Choose $\varepsilon > 0$. Can we

get

$$|x_n - L| < \varepsilon \; ?$$

$$|x_n - L| = |x_n - x_{n_k} + x_{n_k} - L|$$

$$\leq |x_n - x_{n_k}| + |x_{n_k} - L|$$

(triangle inequality)

Choose $K \in \mathbb{N}$ such that

$$|x_{n_k} - L| < \frac{\varepsilon}{2} \quad \forall\, k \geq K.$$

Choose $N_1 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \text{ for all}$$

$$n, m \geq N_1.$$

Let $N = \max\{N_1, n_K\}$

Since $n_k > n_K$ for all

$k \geq K$ (definition of subsequence),

we have

$$|x_n - L| \leq |x_n - x_{n_k}| + |x_{n_k} - L|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all indices greater than N.

$\square$

Equivalent Properties

(see problem 2.6.6)

The following properties are equivalent

1) Monotone Convergence Theorem

2) Nested Interval Property

3) Axiom of Completeness

4) The theorem we just proved – Cauchy Criterion

5) Bolzano – Weierstrass

Infinite Series

Motivation:

$$1 + \underbrace{(-1) + 1}_{0} + \underbrace{(-1) + 1}_{0} + \underbrace{(-1) + 1}_{0} \cdots$$

$$= 1$$

$$\underbrace{(1 + (-1))}_{0} + \underbrace{(1 + (-1))}_{0} + \underbrace{(1 + (-1))}_{0} + \cdots$$

$$= 0 \qquad \text{So } 1 = 0$$

This can't be right.

Definition partial sums

Given a sequence $(a_n)_{n\in\mathbb{N}}$ which we want to "add up."

Start with

$$\sum_{n=1}^{k} a_n = S_k$$

the k^{th} partial sum

Example 1: (geometric series)

Compute partial sums for

$a_n = r^n$ for some

real number r.

$$S_k = \sum_{n=1}^{k} a_n = \sum_{n=1}^{k} r^n$$

$$= r + r^2 + r^3 + \cdots + r^k$$

$$S_k = r + r^2 + r^3 + \cdots + r^k$$

$$S_{k+1} = r + \left(r^2 + r^3 + \cdots + r^k + r^{k+1}\right)$$

$$= r + r\left(r + r^2 + \cdots + r^{k-1} + r^k\right)$$

$$= \boxed{r + r\left(S_k\right)}$$

But also

$$S_{k+1} = \left(r + r^2 + r^3 + \cdots + r^k\right) + r^{k+1}$$

$$= \boxed{S_k + r^{k+1}}$$

So then

$$r + r(S_k) = S_k + r^{k+1}$$

$$r + r(S_k) = S_k + r^{k+1}$$

Solve for S_k

$$S_k - r(S_k) = r - r^{k+1}$$

So

$$S_k = \frac{r - r^{k+1}}{1 - r}$$

$(r \neq 1)$

But if $r = 1$, $a_n = r^n = 1^n = 1$

$S_k = k$ in this case

Convergence resolved

Define $\sum_{n=1}^{\infty} a_n$ for a given sequence $(a_n)_{n\in\mathbb{N}}$ as

$$\boxed{\lim_{k\to\infty}\left(\sum_{n=1}^{k} a_n\right) = \lim_{k\to\infty} S_k}$$

provided the limit exists!

If the limit exists, the series converges. If not, the series diverges.

Example 2: (geometric + motivating example)

For geometric series,

$$S_k = \frac{r - r^{k+1}}{1-r} = \left(\frac{r}{1-r}\right)\left(1 - r^k\right)$$

take limit as $k \to \infty$.

When does it exist?

All that matters is

$\lim_{k\to\infty} r^k$. Dependent on r!

If $r > 1$ $\lim_{k \to \infty} r^k = \infty$, so the series diverges.

If $r \leq -1$ $\lim_{k \to \infty} r^k$ does not exist, so the series diverges.

If $-1 < r < 1$ $\lim_{k \to \infty} r^k = 0$,

and so the series converges to $\frac{r}{1-r} = \lim_{k \to \infty} S_k$.

If $r=1$, $S_k = k$,

$\lim_{k \to \infty} k = \infty$, so the series diverges.

For $a_n = (-1)^{n+1}$

$$(1, -1, 1, -1, \text{---})$$

$$S_{2k} = 0$$

$$S_{2k+1} = 1$$

so the sequence of partial sums is

$$(1, 0, 1, 0, \text{---})$$

which diverges, so the series diverges.