

Announcements

1) HW U up, due next
week

2) One last candidate
tomorrow

2-3 CB 2070

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Theorem : If $X = \mathbb{R}$ with
the metric $d(x, y) = |x - y|$,
then \mathbb{R} is complete.

Proof Let $(x_n)_{n \in \mathbb{N}}$ be a
Cauchy sequence in \mathbb{R}

We know from last class
that all Cauchy sequences
are bounded.

By Bolzano-Weierstrass,

the sequence $(x_n)_{n \in \mathbb{N}}$

admits a convergent

subsequence $(x_{n_k})_{k \in \mathbb{N}}$.

Call the limit of the

subsequence L .

Choose $\varepsilon > 0$. Can we

get

$$|x_n - L| < \varepsilon ?$$

$$\begin{aligned}
 |x_n - L| &= |x_n - x_{n_k} + x_{n_k} - L| \\
 &\leq |x_n - x_{n_k}| + |x_{n_k} - L| \\
 &\quad (\text{triangle inequality})
 \end{aligned}$$

Choose $K \in \mathbb{N}$ such that

$$|x_{n_k} - L| < \frac{\varepsilon}{2} \quad \forall k \geq K.$$

Choose $N_1 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \text{for all}$$

$$n, m \geq N_1.$$

Let $N = \max\{N_1, n_K\}$

Since $n_K > n_K$ for all

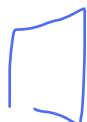
$k \geq K$ (definition of subsequence),

we have

$$|x_n - L| \leq |x_n - x_{n_K}| + |x_{n_K} - L|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all indices greater than N .



Equivalent Properties

(see problem 2.6.6)

The following properties are equivalent

1) Monotone Convergence Theorem

2) Nested Interval Property

3) Axiom of Completeness

4) The theorem we just proved - Cauchy Criterion

5) Bolzano - Weierstrass

Infinite Series

Motivation:

$$1 + \underbrace{((-1) + 1)}_0 + \underbrace{((-1) + 1)}_0 + \underbrace{((-1) + 1)}_0 + \dots \\ = 1$$

$$\underbrace{(1 + (-1))}_0 + \underbrace{(1 + (-1))}_0 + \underbrace{(1 + (-1))}_0 + \dots \\ = 0 \quad \text{So} \quad 1 = 0$$

This can't be right.

Definition partial sums

Given a sequence $(a_n)_{n \in \mathbb{N}}$
which we want to "add up."

Start with

$$\sum_{n=1}^k a_n = S_k$$

the k^{th} partial sum

Example 1: (geometric series)

Compute partial sums for

$$a_n = r^n \text{ for some}$$

real number r .

$$S_k = \sum_{n=1}^k a_n = \sum_{n=1}^k r^n$$

$$= r + r^2 + r^3 + \dots + r^k$$

$$S_k = r + r^2 + r^3 + \dots + r^k$$

$$S_{k+1} = r + \left(r + r^2 + r^3 + \dots + r^k + r^{k+1} \right)$$

$$= r + r(r + r^2 + \dots + r^{k-1} + r^k)$$

$$= \boxed{r + r(S_k)}$$

But also

$$S_{k+1} = \left(r + r^2 + r^3 + \dots + r^k \right) + r^{k+1}$$

$$= \boxed{S_k + r^{k+1}}$$

So then

$$r + r(S_k) = S_k + r^{k+1}$$

$$r + r(S_k) = S_k + r^{k+1}$$

Solve for S_k

$$S_k - r(S_k) = r - r^{k+1}$$

so

$$S_k = \frac{r - r^{k+1}}{1 - r}$$

($r \neq 1$)

But if $r = 1$, $a_n = r^n = 1^n = 1$

$S_k = k$ in this case

Convergence resolved

Define $\sum_{n=1}^{\infty} a_n$ for a given sequence $(a_n)_{n \in \mathbb{N}}$ as

$$\lim_{K \rightarrow \infty} \left(\sum_{n=1}^K a_n \right) = \lim_{K \rightarrow \infty} S_K$$

provided the limit exists!

If the limit exists, the series converges. If not, the series diverges.

Example 2: (geometric + motivation)
example)

For geometric series,

$$S_k = \frac{r - r^{k+1}}{1 - r} = \left(\frac{r}{1-r}\right) (1 - r^k)$$

take limit as $k \rightarrow \infty$.

when does it exist?

All that matters is

$\lim_{k \rightarrow \infty} r^k$. Dependent
on r !

If $r > 1$ $\lim_{k \rightarrow \infty} r^k = \infty$, so

the series diverges.

If $r \leq -1$ $\lim_{k \rightarrow \infty} r^k$ does not

exist, so the series diverges.

If $-1 < r < 1$ $\lim_{k \rightarrow \infty} r^k = 0$,

and so the series converges

$$\text{to } \frac{r}{1-r} = \lim_{k \rightarrow \infty} S_k.$$

If $r=1$, $S_n = n$,

$\lim_{n \rightarrow \infty} n = \infty$, so the

series diverges.

For $a_n = (-1)^{n+1}$

$$(1, -1, 1, -1, \dots)$$

$$S_{2k} = 0$$

$$S_{2k+1} = 1$$

So the sequence of partial sums is

$$(1, 0, 1, 0, \dots)$$

which diverges, so

the series diverges.